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Seiberg-Witten Invariants and Rationality of Complex Surfaces

Christian Okonek* Andrei Teleman*

0 Introduction

Recently, Seiberg and Witten introduced new differential invariants for 4-manifolds, which are defined by counting solutions of the so called *monopole equations*, a system of non-linear differential equations of Yang-Mills-Higgs type [18].

The new invariants are expected to be equivalent to the Donaldson polynomial invariants, and they have already found important applications [15].

The purpose of this paper is:

- to explain the Seiberg-Witten invariants
- to show that — on a Kähler surface — the solutions of the monopole equations can be interpreted as algebraic objects, namely effective divisors
- to give — as an application — a short selfcontained proof for the fact that rationality of complex surfaces is a C^∞ -property.

1 Spin^c -structures and the monopole equation

Definition 1.1 [1], [11] *The group $\text{Spin}^c(n) := \text{Spin}(n) \times_{\mathbb{Z}_2} S^1$ is called the complex spinor group.*

For the case $n = 4$, there is a natural identification

$$\text{Spin}^c(4) = \{(A, B) \in \text{U}(2) \times \text{U}(2) \mid \det A = \det B\} .$$

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The following diagram summarizes some of the basic relations of $\text{Spin}^c(4)$ to other groups:

$$\begin{array}{ccccc}
& & & \text{U}(2) & \\
& & & \swarrow l & \searrow i \\
S^1 & \longrightarrow & \text{Spin}^c(4) & \longrightarrow & \text{SO}(4) \\
(\cdot)^2 \downarrow & \swarrow \det & \downarrow & \xrightarrow{\text{ad}} & \downarrow (\lambda^+, \lambda^-) \\
S^1 & & \text{U}(2) \times \text{U}(2) & & \text{SO}(3) \times \text{SO}(3)
\end{array}$$

Here $l : \text{U}(2) \longrightarrow \text{Spin}^c(4)$ is the canonical lifting of the homomorphism $i \times \det : \text{U}(2) \longrightarrow \text{SO}(4) \times S^1$ [11], and acts by the formula $\text{U}(2) \ni a \longmapsto \left(\begin{pmatrix} \text{id} & 0 \\ 0 & \det a \end{pmatrix}, a \right) \in \text{Spin}^c(4)$. $\lambda^\pm : \text{SO}(4) \longrightarrow \text{SO}(3)$ are the maps induced by the two projections of $\text{Spin}(4) = \text{SU}(2)^+ \times \text{SU}(2)^-$ onto the factors.

Let X be a closed, oriented *simply connected* 4-manifold, Λ^p the bundle of p -forms on X , and $A^p := A^0(X, \Lambda^p)$ the space of sections in this bundle. Let g be a Riemannian metric on X , denote by P the associated principal $\text{SO}(4)$ -bundle, and by P^\pm the $\text{SO}(3)$ -bundles induced via the morphisms λ^\pm . The real 3-vector bundles $\Lambda_\pm^2 := P^\pm \times_{\text{SO}(3)} \mathbb{R}^3$ can be identified with the bundles of (anti)self-dual 2-forms, hence there is an orthogonal splitting $\Lambda^2 = \Lambda_+^2 \oplus \Lambda_-^2$.

Lemma 1.2 [10] *Given $c \in H^2(X, \mathbb{Z})$ with $w_2(X) \equiv \bar{c} \pmod{2}$ there exists a unique $\text{Spin}^c(4)$ -bundle \hat{P}_c with $P \simeq \hat{P}_c / S^1$, and $c_1(\det(\hat{P}_c)) = c$.*

We denote by Σ_c^\pm the induced $\text{U}(2)$ -vector bundles, and we put $\Sigma_c := \Sigma_c^+ \oplus \Sigma_c^-$.

Lemma 1.3 [1], [11] *The choice of a $\text{Spin}^c(4)$ -lift \hat{P}_c of P induces an isomorphism*

$$\gamma_+ : \Lambda^1 \otimes \mathbb{C} \longrightarrow \text{Hom}_{\mathbb{C}}(\Sigma_c^+, \Sigma_c^-)$$

satisfying the identity $\gamma_+(u)^ \gamma_+(v) + \gamma_+(v)^* \gamma_+(u) = 2g(u, v) \text{id}_{\Sigma_c^+}$ for real cotangent vectors $u, v \in \Lambda^1$.*

We define the homomorphisms $\gamma : \Lambda^1 \longrightarrow \text{End}_0(\Sigma_c)$, $\Gamma : \Lambda^2 \longrightarrow \text{End}_0(\Sigma_c)$ by

$$\gamma(u) := \begin{pmatrix} 0 & -\gamma_+(u)^* \\ \gamma_+(u) & 0 \end{pmatrix}$$

$$\Gamma(u \wedge v) := \frac{1}{2}[\gamma(u), \gamma(v)] ,$$

and we denote by the same symbols also their \mathbb{C} -linear extensions $\Lambda^1 \otimes \mathbb{C} \longrightarrow \text{End}_0(\Sigma_c)$, and $\Lambda^2 \otimes \mathbb{C} \longrightarrow \text{End}_0(\Sigma_c)$. The homomorphism γ defines a map $\Lambda^1 \otimes \Sigma_c \longrightarrow \Sigma_c$, called the *Clifford multiplication*. The map Γ identifies the bundles Λ_{\pm}^2 with the bundles of *trace free skew-Hermitian* endomorphisms of Σ_c^{\pm} .

Fix a $\text{Spin}^c(4)$ -bundle \hat{P}_c with $P \simeq \hat{P}_c/S^1$, and let $L_c := \det(\hat{P}_c)$ be the associated S^1 -vector bundle. L_c is the unique unitary line bundle with Chern class c .

Remark 1.4 [11] *The choice of a S^1 -connection a in L_c is equivalent to the choice of a $\text{Spin}^c(4)$ -connection A in \hat{P}_c projecting onto the Levi-Civita connection.*

Definition 1.5 *The composition $\not{D}_a : A^0(\Sigma_c) \xrightarrow{\nabla_A} A^1(\Sigma_c) \xrightarrow{\gamma} A^0(\Sigma_c)$ is called the Dirac operator associated to the connection $a \in \mathcal{A}(L_c)$.*

Notation: Let $\mathcal{A}(L_c)$ be the affine space of S^1 -connections in L_c . For a connection $a \in \mathcal{A}(L_c)$, we denote by $F_a \in A^2(\text{ad}(L_c)) = iA^2$ its curvature, and by $F_a^{\pm} \in iA_{\pm}^2$ the components of F_a with respect to the orthogonal splitting $A^2 = A_+^2 \oplus A_-^2$. Every spinor $\Psi \in A^0(\Sigma_c^+)$ has a conjugate $\bar{\Psi} \in A^0(\bar{\Sigma}_c^+)$, and we can interpret $\Psi \otimes \bar{\Psi}$ as a Hermitian endomorphism of Σ_c^+ . Let $(\Psi \otimes \bar{\Psi})_0 \in A^0(\text{End}_0(\Sigma_c^+))$ denote the trace-free component of it.

The *monopole equations* for a pair $(a, \Psi) \in \mathcal{A}(L_c) \times A^0(\Sigma_c^+)$ are the equations [18]:

$$\begin{cases} \not{D}_a \Psi = 0 \\ \Gamma(F_a^+) = 2(\Psi \otimes \bar{\Psi})_0 \end{cases} \quad (SW)$$

Proposition 1.6 (The Weitzenböck formula [11]). *Let s be the scalar curvature of (X, g) . Fix a $\text{Spin}^c(4)$ -structure on X , and choose a S^1 -connection $a \in \mathcal{A}(L_c)$. Then the following identity holds on $A^0(\Sigma_c)$:*

$$\not{D}_a^2 = \nabla_A^* \nabla_A + \frac{1}{2} \Gamma(F_a) + \frac{s}{4} \text{id}_{\Sigma_c} .$$

Corollary 1.7 *Let $\Psi \in A^0(\Sigma_c^+)$. Then*

$$\| \not{D}_a \Psi \|^2 + \frac{1}{2} \left\| \frac{1}{2} \Gamma(F_a^+) - (\Psi \bar{\Psi})_0 \right\|^2 = \| \nabla_A \Psi \|^2 + \frac{1}{8} \| F_a^+ \|^2 + \frac{1}{4} \| \Psi \|^4 + \frac{1}{4} \int_X s |\Psi|^2.$$

Proof: By the Weitzenböck formula we have

$$(\not{D}_a^2 \Psi, \Psi) = (\nabla_A^* \nabla_A \Psi, \Psi) + \frac{1}{2} (\Gamma(F_a^+)(\Psi), \Psi) + \frac{s}{4} (\Psi, \Psi),$$

since $\Gamma(F_a^-)$ vanishes on Σ_c^+ ; integration over X yields:

$$\begin{aligned} \| \not{D}_a \Psi \|^2 + \frac{1}{2} \left\| \frac{1}{2} \Gamma(F_a^+) - (\Psi \bar{\Psi})_0 \right\|^2 &= \int_X (\not{D}_a^2 \Psi, \Psi) + \frac{1}{2} \int_X \left| \frac{1}{2} \Gamma(F_a^+) - (\Psi \bar{\Psi})_0 \right|^2 = \\ &= \| \nabla_A \Psi \|^2 + \frac{1}{2} \int_X (\Gamma(F_a^+), (\Psi \bar{\Psi})_0) + \frac{1}{4} \int_X \lim_{its} s |\Psi|^2 + \\ &\quad + \frac{1}{2} \int_X \frac{1}{4} |\Gamma(F_a^+)|^2 - \frac{1}{2} \int_X (\Gamma(F_a^+), (\Psi \bar{\Psi})_0) + \frac{1}{4} \| \Psi \|^4. \end{aligned}$$

Remark 1.8 [18] *If $s \geq 0$ on X , then the only solutions (a, Ψ) of (SW) are pairs $(a, 0)$ with $F_a^+ = 0$.*

2 Seiberg-Witten Invariants

The gauge group $\mathcal{G} := \mathcal{C}^\infty(X, S^1)$ in the Seiberg-Witten theory is *abelian* and acts on $\mathcal{A}(L_c) \times A^0(\Sigma_c^+)$ by $(a, \Psi) \cdot f := (a + f^{-1} df, f^{-1} \Psi)$, letting invariant the set of solutions of the equations (SW). We denote by $\mathcal{W}_X^g(c)$ the moduli space of solutions of the Seiberg-Witten equations, modulo gauge equivalence. A standard technique provides a natural structure of finite dimensional *real analytic* space in $\mathcal{W}_X^g(c)$ [6], [5], [16]. The *expected dimension* of this moduli space is

$$w_c = \frac{1}{4}(c^2 - 2e(X) - 3\sigma(X)),$$

where $e(X)$ and $\sigma(X)$ stand for the Euler characteristic and the signature of the oriented manifold X . A solution (a, Ψ) is reducible (has nontrivial stabilizer) if and only if $\Psi = 0$, and then the connection a must be anti-selfdual. We say that the metric g is *c-good* if the g -harmonic representative of the de Rham cohomology class c_{DR} is *not* anti-selfdual. If g is *c-good*, then $\mathcal{W}_X^g(c)$ consists only of irreducible orbits.

Using the same technique as in Yang-Mills theory ([6], [5]), one defines a *gauge invariant perturbation* of the Seiberg-Witten equations in order to get smooth moduli spaces of the expected dimension. For a selfdual form $\mu \in A_+^2$ we denote by $\mathcal{W}_X^{g,\mu}(c)$ the moduli space of solutions of the perturbed Seiberg-Witten equations

$$\begin{cases} \not{D}_a \Psi &= 0 \\ \Gamma(F_a^+ + i\mu) &= 2(\Psi \otimes \bar{\Psi})_0 \end{cases} \quad (SW_\mu)$$

We refer to [15] for the following

Lemma 2.1

1. For every $\mu \in A_+^2$, the moduli space $\mathcal{W}_X^{g,\mu}(c)$ is compact.
2. There is a dense, second category set of perturbations $\mu \in A_+^2$, for which the irreducible part $\mathcal{W}_X^{g,\mu}(c)^*$ of $\mathcal{W}_X^{g,\mu}(c)$ is smooth and has the expected dimension.
3. If g is c -good, and μ is small enough in the L^2 topology, then $\mathcal{W}_X^{g,\mu}(c)$ consists only of irreducible orbits, i.e. $\mathcal{W}_X^{g,\mu}(c) = \mathcal{W}_X^{g,\mu}(c)^*$.
4. Let g_0 and g_1 be c -good metrics which can be connected by a smooth path of c -good metrics, and let $\varepsilon_i > 0$ be small enough such that $\mathcal{W}_X^{g_i,\mu_i}(c) = \mathcal{W}_X^{g_i,\mu_i}(c)^*$ for all perturbations μ_i with $\|\mu_i\| < \varepsilon_i$. Then any two moduli spaces $\mathcal{W}_X^{g_i,\mu_i}(c)$, $i = 0, 1$, with $\|\mu_i\| < \varepsilon_i$, which are smooth and have the expected dimension, are cobordant.

The first assertion is a simple consequence of the Weitzenböck formula and of the Maximum Principle. The other three assertions follow as in Donaldson theory by the Sard theorem for smooth Fredholm maps, and by transversality arguments [5]. Note that in 4. we mean cobordism between *non-oriented* compact smooth manifolds. A more delicate analysis of the monopole equations [18] shows that, in fact, the moduli spaces $\mathcal{W}_X^{g,\mu}(c)^*$ come with natural orientations, as soon as they are smooth and have the expected dimension, and that the conclusion in 4. holds for the oriented moduli spaces.

The Seiberg-Witten theory provides strong differentiable invariants using only moduli spaces of dimension 0. Let c be an integral lift of $w_2(X)$, with $w_c = 0$, i.e.

$$c^2 = 2e(X) + 3\sigma(X) .$$

Such a lift is called an *almost canonical class*, since the condition $w_c = 0$ is equivalent to the existence of an almost complex structure on X with first Chern class c [10], [16].

Now fix an almost canonical class c , choose a c -good metric g , and a small, sufficiently general perturbation μ . Then $\mathcal{W}_X^{g,\mu}(c) = \mathcal{W}_X^{g,\mu}(c)^*$ is compact, smooth of the expected dimension 0, and its bordism class is independent of μ . Let $n_c^g := |\mathcal{W}_X^{g,\mu}(c)| \bmod 2$ be the number of points modulo 2 of this moduli space. Lemma 2.1 implies that n_c^g is also independent of g if any two c -good metrics can be connected by a smooth 1-parameter family of c -good metrics.

The numbers $n_c := n_c^g$ associated to such almost canonical classes are called the mod 2-Seiberg-Witten invariants, and the classes c with $n_c \neq 0$ are then called mod 2-Seiberg-Witten classes of index 0. By definition they are differentiable invariants, in the following sense: If $f : X' \rightarrow X$ is an orientation-preserving diffeomorphism, and for an almost canonical class c of X the Seiberg-Witten invariant n_c is well defined, then $f^*(c)$ has the same property, and $n_{f^*(c)} = n_c$.

Remark 2.2 *Let c be an almost canonical class of X .*

1. *If $c^2 \geq 0$ and $c_{\text{DR}} \neq 0$, then any Riemannian metric on X is c -good.*
2. *If $b_2^+ \geq 2$, then any two c -good metrics can be connected by a smooth path of c -good metrics.*

Therefore, if one of the two conditions above is satisfied, then the mod 2-Seiberg-Witten invariant n_c is well-defined.

In the case $b_2^+ = 1$, invariants can still be defined, but the dependence of n_c^g on the metric g must be taken into account: In the real vector space $H_{\text{DR}}^2(X)$, consider the positive cone

$$\mathcal{K} = \{u \in H_{\text{DR}}^2(X) \mid u^2 > 0\} .$$

Fix a *non-vanishing* cohomology class $k \in H_{\text{DR}}^2(X)$ with $k^2 \geq 0$. The cone \mathcal{K} splits as the disjoint union of its connected components \mathcal{K}_{\pm} , where

$$\mathcal{K}_{\pm} := \{u \in \mathcal{K} \mid \pm u \cdot k > 0\} .$$

If c is an almost canonical class, let c^{\perp} be the hyperplane

$$c^{\perp} := \{u \in H_{\text{DR}}^2(X) \mid c \cdot u = 0\}$$

If c^\perp meets \mathcal{K}_+ , then the intersection $c^\perp \cap \mathcal{K}_+$ is called the *wall* of type c , and the two components of $\mathcal{K}_+ \setminus c^\perp$ are called *chambers* of type c . For every Riemannian metric g on X , let ω_g be a generator of the real line of g -harmonic selfdual 2-forms, such that $[\omega_g] \in \mathcal{K}_+$. Then the ray $\mathbb{R}_{>0}[\omega_g] \subset \mathcal{K}_+$ depends smoothly on the metric g . The property of a metric to be c -good has the following simple geometric interpretation:

Remark 2.3

Suppose $b_2^+(X) = 1$. Then:

1. The metric g is c -good iff the ray $\mathbb{R}_{>0}[\omega_g]$ does not lie in the wall $c^\perp \cap \mathcal{K}_+$.
2. If g_0 , and g_1 are c -good metrics, then $n_c^{g_0} = n_c^{g_1}$ iff the two rays $\mathbb{R}_{>0}[\omega_{g_i}]$ belong to the same chamber of type c .

The first assertion follows immediately from the definition. The second needs a careful analysis of a 1-parameter family of 0-dimensional smooth moduli spaces $\mathcal{W}_X^{gt}(c)$ around the value of the parameter t for which the ray $\mathbb{R}_{>0}[\omega_{g_t}]$ crosses the wall $c^\perp \cap \mathcal{K}_+$ (see [18], [15]).

3 Monopoles on Kähler surfaces

Let (X, J, g) be an almost complex 4-manifold endowed with a Hermitian metric g . The almost complex structure J defines a reduction of the structure group of the tangent bundle T_X of X from $\mathrm{SO}(4)$ to $\mathrm{U}(2)$. In particular, we get a *canonical* $\mathrm{Spin}^c(4)$ -structure on X via the canonical lifting $l : \mathrm{U}(2) \longrightarrow \mathrm{Spin}^c(4)$ [11]. Let ω_g be the Kähler form of g .

Lemma 3.1 [11] *The canonical Spin^c -structure of an almost complex Hermitian 4-manifold has the following properties:*

1. There are canonical identifications $\Sigma^+ = \Lambda^{00} \oplus \Lambda^{02}$, $\Sigma^- = \Lambda^{01}$.
2. Via these identifications, the map $\Gamma : \Lambda_+^2 \otimes \mathbb{C} \longrightarrow \mathrm{End}_0(\Sigma^+)$ is given by:

$$\Lambda^{20} \oplus \Lambda^{02} \oplus \Lambda^{00} \omega_g \ni (\lambda^{20}, \lambda^{02}, f\omega_g) \mapsto 2 \begin{bmatrix} -if & - * (\lambda^{20} \wedge \cdot) \\ \lambda^{02} \wedge \cdot & if \end{bmatrix} \in \mathrm{End}_0(\Lambda^{00} \oplus \Lambda^{02}).$$

Suppose now that (X, J, g) is a Kähler surface. This means that J is integrable, and ω_g is closed (or equivalently, J is Levi-Civita parallel). In particular the holonomy group of the Levi-Civita connection also reduces to $\mathrm{U}(2)$, and the splittings $\Lambda^p \otimes \mathbb{C} = \bigoplus_{i+j=p} \Lambda^{ij}$ are Levi-Civita parallel. We get a

U(2)-connection in the holomorphic tangent bundle $\mathcal{T}_X = T_X^{10} \simeq \Lambda^{01}$, which coincides with the *Chern connection* of this bundle, i.e. with the unique connection compatible with the holomorphic structure and the Hermitian metric. The induced connection c_0 in the line bundle $K_X^\vee = \det(\mathcal{T}_X) \simeq \Lambda^{02}$ also coincides with the Chern connection of this Hermitian holomorphic line bundle.

Every other $\text{Spin}^c(4)$ structure $\hat{P}_c \longrightarrow P$ on (X, g) has as spinor bundle

$$\Sigma_c = \Sigma \otimes M ,$$

where M is a differentiable S^1 -bundle with $2c_1(M) + c_1(K_X^\vee) = c$. (For a simply connected manifold X , M is well defined up to isomorphism by this condition.) S^1 -connections in $\det(\Sigma_c^\pm) = K_X^\vee \otimes M^{\otimes 2}$ correspond to S^1 -connections in M . Given $b \in \mathcal{A}(M)$, the curvature of the corresponding connection $a \in \mathcal{A}(K_X^\vee \otimes M^{\otimes 2})$ is $F_a = F_{c_0} + 2F_b$.

A half-spinor $\Psi \in A^0(\Sigma^+ \otimes M)$ can be written as

$$\Psi = \varphi + \alpha , \quad \varphi \in A^0(M) , \quad \alpha \in A^{02}(M) .$$

We put $J(M) := c_1(\Sigma^+ \otimes M) \cup [\omega_g]$.

Proposition 3.2 *Let (X, g) be a Kähler surface with Chern connection c_0 in K_X^\vee , M a differentiable S^1 -bundle with $J(M) < 0$. A pair $(b, \varphi + \alpha) \in \mathcal{A}(M) \times (A^0(M) \oplus A^{02}(M))$ solves the monopole equations iff:*

$$\begin{aligned} F_b^{20} &= F_b^{02} = 0 \\ \alpha &= 0 , \quad \bar{\partial}_b(\varphi) = 0 \\ i\Lambda F_b + \frac{1}{2}\varphi\bar{\varphi} + \frac{s}{2} &= 0 . \end{aligned} \tag{*}$$

Proof: The pair $(b, \varphi + \alpha)$ solves the equations (SW) iff the corresponding pair $(a, \varphi + \alpha)$ satisfies

$$\begin{aligned} F_a^{20} &= -\varphi \otimes \bar{\alpha} \\ F_a^{02} &= \alpha \otimes \bar{\varphi} \\ \bar{\partial}_b(\varphi) &= i\Lambda\bar{\partial}_b(\alpha) \\ i\Lambda F_a &= -(\varphi\bar{\varphi} - *(\alpha \wedge \bar{\alpha})) . \end{aligned}$$

By Corollary 1.7 it follows that $(b, \varphi + \alpha)$ solves (SW) iff $(b, \varphi - \alpha)$ does (Witten's trick). Therefore $\varphi \otimes \bar{\alpha} = \alpha \otimes \bar{\varphi} = 0$, hence $F_a^{20} = F_a^{02} = 0$, and φ

or α must vanish. Integrating the equation $i\Lambda F_a = -(\varphi\bar{\varphi} - *(\alpha \wedge \bar{\alpha}))$ over X , we find:

$$J(M) = (2c_1(M) - c_1(K_X)) \cup [\omega_g] = \int_X \frac{i}{2\pi} F_a \wedge \omega_g = \frac{1}{8\pi} \int_X (-|\varphi|^2 + |\alpha|^2) ,$$

hence $\alpha = 0$ if $J(M) < 0$. ■

The above proposition must be interpreted as follows: If $J(M) < 0$, then the solutions of the monopole equations (*SW*) are the pairs $(b, \varphi) \in \mathcal{A}(M) \times A^0(M)$, such that b is the Chern connection of a holomorphic structure in M , φ is a holomorphic section, and the mean curvature $i\Lambda F_b$ of b satisfies the *generalized vortex equation* [16], [3], [4], [9]

$$i\Lambda F_b + \frac{1}{2}\varphi\bar{\varphi} + \frac{s}{2} = 0 . \quad (V_s)$$

Moreover, every *infinitesimal deformation* of a solution of the form (b, φ) , $\varphi \neq 0$ of the monopole equation still vanishes in the α -direction. Therefore $\mathcal{W}_X^g(c)$ can be identified (as *real analytic space*) with the moduli space of pairs (b, φ) satisfying the above conditions, modulo the gauge group $\mathcal{C}^\infty(X, S^1)$ of unitary automorphisms of M . Under the assumption $J(M) < 0$, the action of the gauge group is free on the space of solutions, because any solution (b, φ) has a non-vanishing section φ .

Alternatively, let \mathcal{M} be a holomorphic line bundle with differentiable support M , and φ a holomorphic section of \mathcal{M} . For a Hermitian metric h in \mathcal{M} , we denote by F_h the curvature of the associated Chern connection, and we consider the following equation for h :

$$i\Lambda F_h + \frac{1}{2}\varphi\bar{\varphi}^h + \frac{s}{2} = 0 . \quad (V'_s)$$

Standard arguments (see for instance [16], [9]) show that the problem of classifying the solutions (b, φ) of $(*)$ modulo *unitary automorphisms of M* is equivalent to the problem of classifying those pairs (\mathcal{M}, φ) modulo *holomorphic isomorphisms*, for which the equation (V'_s) has a solution.

Proposition 3.3 *Let (X, g) be a compact Kähler surface, (\mathcal{M}, φ) a holomorphic line bundle with a non-vanishing holomorphic section $\varphi \in H^0(X, \mathcal{M})$. \mathcal{M} admits a metric h satisfying the equation (V'_s) iff*

$$c_1(\mathcal{M}) \cup [\omega_g] < \frac{1}{2}c_1(K_X) \cup [\omega_g] .$$

Proof: (cf. [3]) Fix a background metric h_0 ; any other metric h has the form $h = e^{2u}h_0$, with $u \in A^0$ a smooth function. The vortex equation (V'_s) translates into

$$\Delta u + \frac{1}{2}|\varphi|_{h_0}^2 e^{2u} + (i\Lambda F_{h_0} + \frac{s}{2}) = 0 . \quad (1)$$

Set $q := \int_X (i\Lambda F_{h_0} + \frac{s}{2}) = 2\pi(c_1(\mathcal{M}) - \frac{1}{2}c_1(K_X) \cup [\omega_g])$, and choose $v \in A^0$ with

$$-\Delta v = (i\Lambda F_{h_0} + \frac{s}{2}) - q .$$

Define $w := 2(u - v)$. Then (1) is equivalent to the following equation in w :

$$\Delta w + (|\varphi|_{h_0}^2 e^{2v})e^w + 2q = 0 . \quad (2)$$

Integrating over X , we see that if (2) has solutions, then q must be negative. On the other hand, by a well known result of Kazdan and Warner [3], (2) has a unique solution if $q < 0$. \blacksquare

Theorem 3.4 [18], [16] *Let (X, g) be a simply connected Kähler surface, $c \in H^2(X, \mathbb{Z})$ with $c \equiv c_1(K_X) \bmod 2$, and $\pm c \cup [\omega_g] < 0$.*

1. *If $c \notin \text{NS}(X)$, then $\mathcal{W}_X^g(c) = \emptyset$.*
2. *Suppose $c \in \text{NS}(X)$. Then there is a natural real analytic isomorphism $\mathcal{W}_X^g(c) \simeq \mathbb{P}(H^0(X, \mathcal{M}))$, where \mathcal{M} is the (unique, up to isomorphism) holomorphic line bundle with $c_1(K_X^\vee \otimes \mathcal{M}^{\otimes 2}) = \pm c$.*
3. *$\mathcal{W}_X^g(c)$ is always smooth. Let D be the divisor of a nontrivial section in \mathcal{M} . Then $\mathcal{W}_X^g(c)$ has the expected dimension iff $h^1(\mathcal{O}_X(D)|_D) = 0$.*

4 Rationality of complex surfaces

A compact complex surface is *rational* iff its field of meromorphic functions is isomorphic to $\mathbb{C}(u, v)$. Such a surface is always simply connected and has $b_2^+ = 1$ [2]. The following result has been announced by R. Friedman and Z. Qin [8]. Whereas their proof uses Donaldson theory and vector bundles techniques, our proof uses the new Seiberg-Witten invariants, and our interpretation of these invariants in terms of linear systems.

Theorem 4.1 [17] *A complex surface X which is diffeomorphic to a rational surface is rational.*

Proof: The proof consists of the following three steps:

1. Any rational surface X_0 admits a *Hitchin metric* [12], i.e. a Kähler metric g_0 with positive *total scalar curvature*. This condition can be written as $c_1(K_{X_0}) \cup [\omega_{g_0}] < 0$.

Let c be any integral lift of $w_2(X_0)$, such that g_0 is c -good, i.e. such that the moduli space $\mathcal{W}_{X_0}^{g_0}(c)$ contains no reducible solutions. Since $p_g(X_0) = 0$, c has always type $(1,1)$, and g_0 is c -good iff $c \cup [\omega_{g_0}] \neq 0$.

We assert that $\mathcal{W}_{X_0}^{g_0}(c)$ is then empty, and in particular, all Seiberg-Witten invariants $n_c^{g_0}$ computed with respect to this metric vanish.

Indeed, let \mathcal{M} be the holomorphic line bundle defined in Theorem 3.4. If the moduli space $\mathbb{P}(H^0(X_0, \mathcal{M}))$ was not empty, then

$$c_1(\mathcal{M}) \cup [\omega_{g_0}] \geq 0 . \quad (1)$$

But we have

$$0 > \pm c \cup [\omega_{g_0}] = (2c_1(\mathcal{M}) - c_1(K_{X_0})) \cup [\omega_{g_0}] ,$$

hence, by (1)

$$0 \leq 2c_1(\mathcal{M}) \cup [\omega_{g_0}] < c_1(K_{X_0}) \cup [\omega_{g_0}] ,$$

which contradicts the assumption on the total scalar curvature of g_0 .

2. Let now X be a simply connected projective surface with $\text{kod}(X) > 0$. We may suppose that X is the blow up in k *distinct* points of its minimal model X_{\min} . Denote by $\sigma : X \longrightarrow X_{\min}$ the contraction to the minimal model, and by $E = \sum_{i=1}^k E_i$ the exceptional divisor. Fix an ample divisor H_{\min} on X_{\min} , set $H_n := \sigma^*(nH_{\min}) - E$, and for $n \gg 0$ choose a Kähler metric g_n on X with $[\omega_{g_n}] = c_1(H_n)$. Given $I \subset \{1, \dots, k\}$, define

$$\begin{aligned} E_I &:= \sum_{i \in I} E_i \\ c_I &:= 2c_1(E_I) - c_1(K_X) \\ \bar{I} &:= \{1, \dots, k\} \setminus I . \end{aligned}$$

Since c_I is an almost canonical class, the expected dimension of the corresponding Seiberg-Witten moduli space is 0. For $n \gg 0$ we get $c_I \cup [\omega_{g_n}] < 0$, and Theorem 3.4 gives

$$\mathcal{W}_X^{g_n}(c_I) \simeq \{E_I\} .$$

Therefore $\mathcal{W}_X^{g_n}(c_I)$ consists of a single smooth point, and

$$n_{c_I}^{g_n} = 1 \bmod 2 . \quad (2)$$

3. Suppose now that there is an orientation-preserving diffeomorphism $f : X \rightarrow X_0$, where X is projective surface with $\text{kod}X \geq 0$. Since X must have $p_g(X) = 0$, and $\pi_1(X) = \{1\}$, it follows that, in fact, $\text{kod}X > 0$. Let $g = f^*(g_0)$ denote the pull-back of a Hitchin metric to X ; clearly

$$n_{c_I}^g = 0 \quad (3)$$

for all $I \subset \{1, \dots, k\}$ such that g is c_I -good.

We will now derive a contradiction in the following way: Using the Enriques-Kodaira classification of surfaces, it is easy to see that the de Rham cohomology class $k_{\min} := \sigma^*(c_{1,\text{DR}}(K_{\min}))$ is non-trivial and satisfies the condition $k_{\min}^2 \geq 0$. Therefore we can consider the upper positive cone

$$\mathcal{K}_+ := \{u \in H_{\text{DR}}^2(X) \mid u^2 > 0, u \cdot k_{\min} > 0\} .$$

Clearly $[\omega_{g_n}]$ belongs to \mathcal{K}_+ . We choose a harmonic g -selfdual form ω_g , with $[\omega_g] \in \mathcal{K}_+$.

Claim: *The rays $\mathbb{R}_{>0}[\omega_g]$ and $\mathbb{R}_{>0}[\omega_{g_n}]$ belong either to the same chamber of type c_I or to the same chamber of type $c_{\bar{I}}$.*

Proof: If not, then, since $c_I \cup [\omega_{g_n}] < 0$, we get $[\omega_g] \cdot c_I \geq 0$ and $[\omega_g] \cdot c_{\bar{I}} \geq 0$. Write

$$[\omega_g] = \sum_{i=1}^k \lambda_i E_i + \sigma^*[\omega] ,$$

with $[\omega] \in H_{\text{DR}}^2(X_{\min})$. Then

$$\begin{aligned} - \sum_{i \in I} \lambda_i + \sum_{j \in \bar{I}} \lambda_j - [\omega] \cdot [K_{\min}] &\geq 0 \\ - \sum_{j \in \bar{I}} \lambda_j + \sum_{i \in I} \lambda_i - [\omega] \cdot [K_{\min}] &\geq 0 . \end{aligned}$$

Adding these inequalities we find $[\omega] \cdot [K_{\min}] \leq 0$. But $[\omega] \cdot [K_{\min}] = [\omega_g] \cdot k_{\min} > 0$, because $[\omega_g] \in \mathcal{K}_+$. This contradiction proves the claim.

It follows that either g and g_n are both c_I -good and $n_{c_I}^g = n_{c_I}^{g_n}$, or g and g_n are both $c_{\bar{I}}$ -good and $n_{c_{\bar{I}}}^g = n_{c_{\bar{I}}}^{g_n}$. This gives now a contradiction with (2) and (3). \blacksquare

Together with the results of Friedman and Morgan [7], we have:

Theorem 4.2 (The Van de Ven conjecture [19]) *The Kodaira dimension of complex surfaces is a C^∞ -invariant.*

Remark: It is possible to couple the Seiberg-Witten equations to connections in unitary bundles. The solutions of these coupled Seiberg-Witten equations over Kähler surfaces again have a purely complex-geometric interpretation [16]: The moduli space of solutions can be identified—via generalized vortex equations—with moduli spaces of stable pairs [13], [4]. This construction could lead to new invariants which might be nontrivial for Kähler surfaces with $p_g = 0$.

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